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Published in:
Mathematical Social Sciences

Publication date:
1994

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Peters, H., Tijs, S. H., & Zarzuelo, J. (1994). A reduced game property for the Kalai-Smorodinsky and egalitarian bargaining solutions. *Mathematical Social Sciences*, 27(1), 11-18.

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A reduced game property for the Kalai–Smorodinsky and egalitarian bargaining solutions

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Communicated by H. Moulin

Received 12 June 1992

Revised 5 July 1993

Abstract

An axiomatic characterization of the n -person Kalai–Smorodinsky bargaining solution is presented, based on a reduced game property. An extension to a large class of solutions including the egalitarian solution is also provided.

Key words: Reduced games; Cooperative games; Kalai–Smorodinsky bargaining solution; Egalitarian solution

1. Introduction

In recent years the concepts of reduced games and corresponding reduced game properties have proved to be fruitful tools in cooperative game theory. Many well-known solution concepts have been characterized with the aid of such concepts.

The general principle is as follows. Given a solution concept for a class of games, a game in that class, and a subset of the set of players involved in that

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game, one defines a new ('reduced') game for that subset of players, depending in some way or other on the outcome assigned to the original game—in particular, the payoffs for the players outside that subset may be specified; then the solution at hand satisfies the 'reduced game property' (or: is 'consistent') if the players in the reduced game obtain the same payoffs as in the original game at the solution outcome.

In the area of transferable (and non-transferable) utility games, by now most accepted solution concepts [(pre)nucleolus, core, Shapley value] have been characterized by appropriate reduced game properties – see Driessen (1991) for a survey. Not all of these properties have an equally natural or intuitive economic interpretation, but they do conform to the general principle formulated above and characterize the solution concept under consideration. Thus, the issue has two sides: Given an appealing reduced game property, are there solutions satisfying it? But also: Given an accepted solution concept, can one find a reduced game property satisfied by it?

Also in the area of bargaining, reduced game properties have been studied – see Thomson (1990) for an overview. In particular, Lensberg (1988) gives a characterization of the Nash bargaining solution based on a very natural reduced game property called 'multilateral stability'. Unfortunately, the Kalai–Smorodinsky bargaining solution (Kalai and Smorodinsky, 1975; Raiffa, 1953) does not satisfy this property. Therefore, in this paper we propose a different reduced game property for bargaining solutions, which is satisfied by the Kalai–Smorodinsky solution. Furthermore, a characterization based on this property is given. The property is perhaps most closely related to Thomson's (1983) axiom of monotonicity with respect to changes in the number of agents.

Section 2 presents the reduced game property and the axiomatic characterization of the Kalai–Smorodinsky solution. Furthermore, by weakening the axioms this characterization is extended to a large class of solutions including the egalitarian solution. Section 3 concludes with a remark on related non-co-operative models.

2. The reduced game property and characterization results

Let M , a finite subset of the natural numbers, denote a set of players. Let \mathbb{R}_+^M denote the Cartesian product of $|M|$ copies of \mathbb{R}_+ indexed by the players in M . A *bargaining game* for M is a subset S of \mathbb{R}_+^M satisfying the following requirements:

- S is non-empty and compact, and contains a strictly positive vector.
- S is *comprehensive*, i.e. $y \in S$ whenever $y \in \mathbb{R}_+^M$ and $y \leq x$ for some $x \in S$.

The interpretation of such a bargaining game is that the players in M try to reach an agreement or *outcome* $x \in S$, giving utility x_i to player $i \in M$. If they fail, each player ends up with zero utility. The conditions imposed here on a bargaining game are standard in axiomatic bargaining theory. Note that there is no convexity requirement on a bargaining game S ; all our results hold without this standard

requirement, but remain valid if it is included. The *disagreement outcome* has been normalized to the origin (see also the remark below concerning the scale invariance axiom).

B^M denotes the set of all bargaining games for M .

Let N be a given set (population) of potential players. N may be a finite but also an infinite subset of the natural numbers. Let

$$B_N := \bigcup_{\emptyset \neq M \subset N, M \text{ is finite}} B^M$$

denote the collection of all bargaining games for finite subsets of N . A (*bargaining*) *solution* on B_N is a function φ on B_N with $\varphi(S) \in S$ for all $S \in B_N$. Two-person bargaining games were introduced by Nash (1950), while the idea to consider a variable number of players, is, in this context, due to Thomson (1983). The axiomatic approach to bargaining implies formulating ‘reasonable’ properties or *axioms* for bargaining solutions, and characterizing solutions by these axioms. Here, we are interested in the *Kalai–Smorodinsky solution* κ defined as follows (cf. Raiffa, 1953; Kalai and Smorodinsky, 1975). For $S \in B^M$ let the *utopia point* $u(S) \in \mathbb{R}_+^M$ be defined by

$$u_i(S) := \max_{x \in S} x_i$$

for all $i \in M$. Then $\kappa(S)$ is the maximal point of S on the line segment connecting $u(S)$ and the origin. Let $W(S) := \{x \in S \mid \text{there is no } y \in S \text{ with } y > x\}$ denote the *weakly Pareto optimal* subset of a bargaining game S . It is straightforward to verify that κ satisfies the following three ‘standard’ axioms for a solution φ on B_N :

Weak Pareto Optimality (WPO): $\varphi(S) \in W(S)$ for all $S \in B_N$.

Anonymity (AN): For every finite $M \subset N$, all $i, j \in M$, and all $S, T \in B^M$ such that T arises from S by interchanging the i th and j th coordinates of the points of S , we have: $\varphi_i(S) = \varphi_j(T)$, $\varphi_j(S) = \varphi_i(T)$, and $\varphi_k(S) = \varphi_k(T)$ for all $k \neq i, j$.

Scale invariance (SI): For every finite subset M of N and every vector $a \in \mathbb{R}_+^M$, we have $\varphi(aS) = a\varphi(S)$, where $(ax)_i := a_i x_i$ for all $x \in \mathbb{R}_+^M$ and $i \in M$, and $aS := \{ax \mid x \in S\}$.

As is well known and obvious, the Kalai–Smorodinsky solution does not satisfy the stronger version of WPO defined by requiring the solution outcome to be not even weakly dominated.

Anonymity requires that the names of the players do not matter.

The usual formulation of Scale Invariance contains a ‘translation invariance’ part: here, this part is implicit by the normalization of the disagreement point to the origin.

For later reference we define the following weakening of SI.

Homogeneity (HOM): For every finite subset M of N and every vector $a \in \mathbb{R}_+^M$ with $a_i = a_j$ for all $i, j \in M$, we have $\varphi(aS) = a\varphi(S)$.

In order to formulate the main axiom we first need to introduce the concept of a reduced game. Let L and M be non-empty finite subsets of N with $L \subset M$. Let $S \in B^M$. For $x \in \mathbb{R}^M$, x_L denotes the vector arising from x by deleting the coordinates in $M \setminus L$, i.e. the projection of x on \mathbb{R}^L . Then S_L denotes the bargaining game $\{x_L \mid x \in S\}$ in B^L . Let $x \in S$, $x \neq 0$, $x_L \neq 0$. Let

$$\lambda(S_L, x_L) := \min\{\lambda \in \mathbb{R}_+ \mid x_L \in \lambda S_L\}.$$

The *reduced game of S with respect to L and x* is the following bargaining game for L :

$$S_L^x := \lambda(S_L, x_L)S_L.$$

Note that x_L is an element of the weakly Pareto optimal subset of S_L^x . The reduced game S_L^x is a multiple of the game the players in L would be able to play if the players in M outside L could be sent off with nothing. This multiple is chosen in such a way that the players outside L may still obtain their payoffs according to the original outcome x , while leaving a weakly Pareto optimal outcome x_L for the players in the reduced game. The reduced game property requires of a solution φ to pick this point x_L in the reduced game if x is chosen in the original game. Thus:

Reduced Game Property (RGP): For all non-empty finite subsets $L \subset M$ of N and all $S \in B^M$: if $\varphi(S)_L \neq 0$, then $\varphi(S_L^{\varphi(S)}) = \varphi(S)_L$.

For a homogeneous solution an interpretation of RGP is as follows. Consider the games S_L as prenegotiations of the coalitions $L \subset M$. RGP then requires that the final solution outcome for the grand coalition be proportional to each of the coalitional prenegotiation outcomes, i.e. that the established power proportions are preserved. This distinguishes RGP from the multilateral stability axiom of Lensberg (1988), where the reduced game is constrained by the payoffs of the ‘deleted’ players. For a given game, recursive application of RGP in fact learns that for each smaller game arising if a subset of the players were absent, the payoffs of the remaining players are in the same fixed proportion. Proportional solutions (Kalai, 1977), which include the egalitarian solution (see below), obviously satisfy RGP. We do not claim that this is a natural requirement in all situations; but then again, we do not claim that the Kalai–Smorodinsky solution or proportional solutions are appropriate in all situations.

It is easy to verify that the Kalai–Smorodinsky solution κ satisfies RGP. This is a direct consequence of the fact that the utopia point of a game S_L is the projection of the utopia point of S . See Fig. 1 for an illustration with $M = \{1, 2, 3\}$ and $L = \{1, 2\}$. Moreover, if the population contains at least three players, the solution is characterized by four of the axioms hitherto defined:

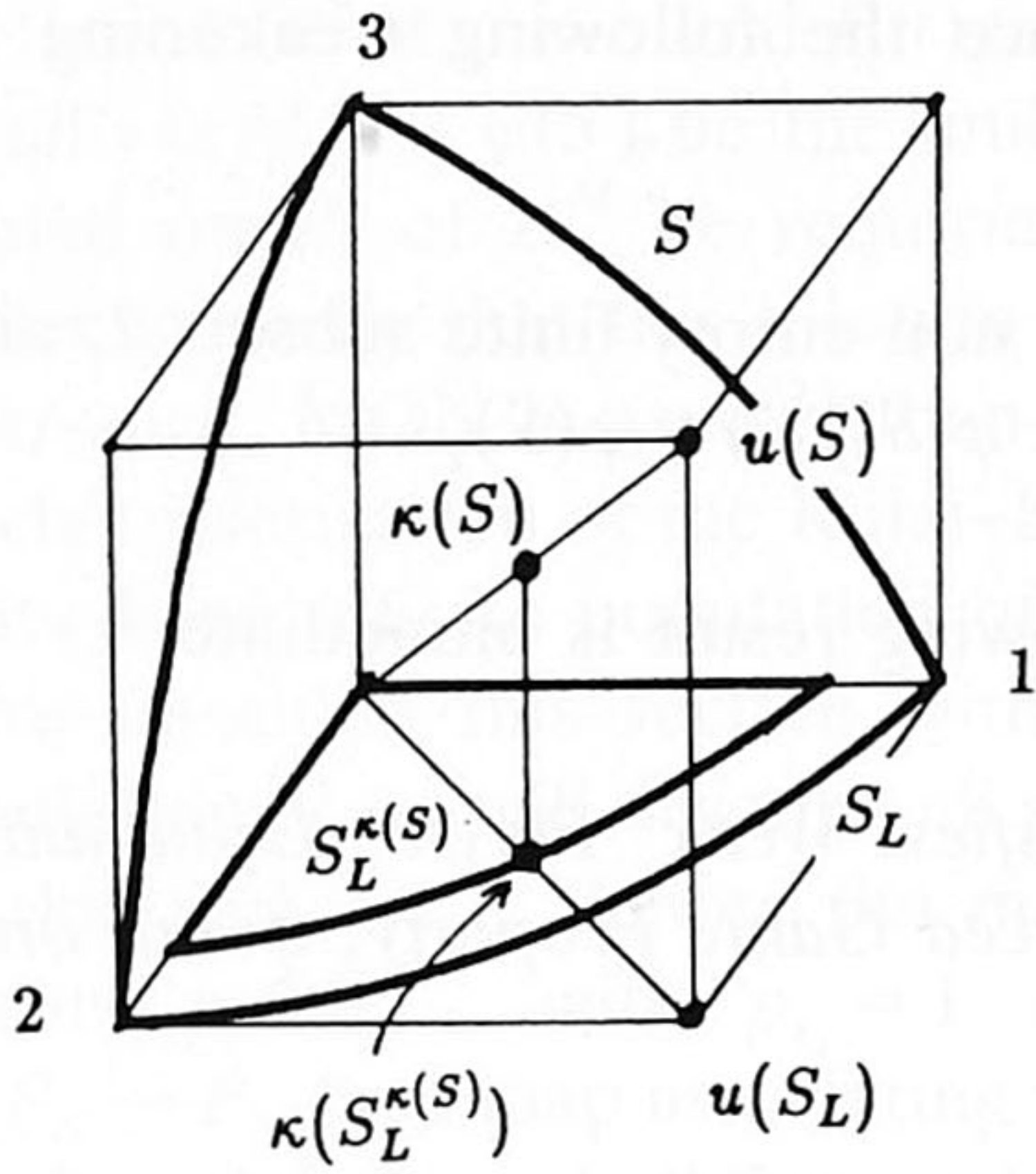


Fig. 1. The Kalai–Smorodinsky solution has the reduced game property.

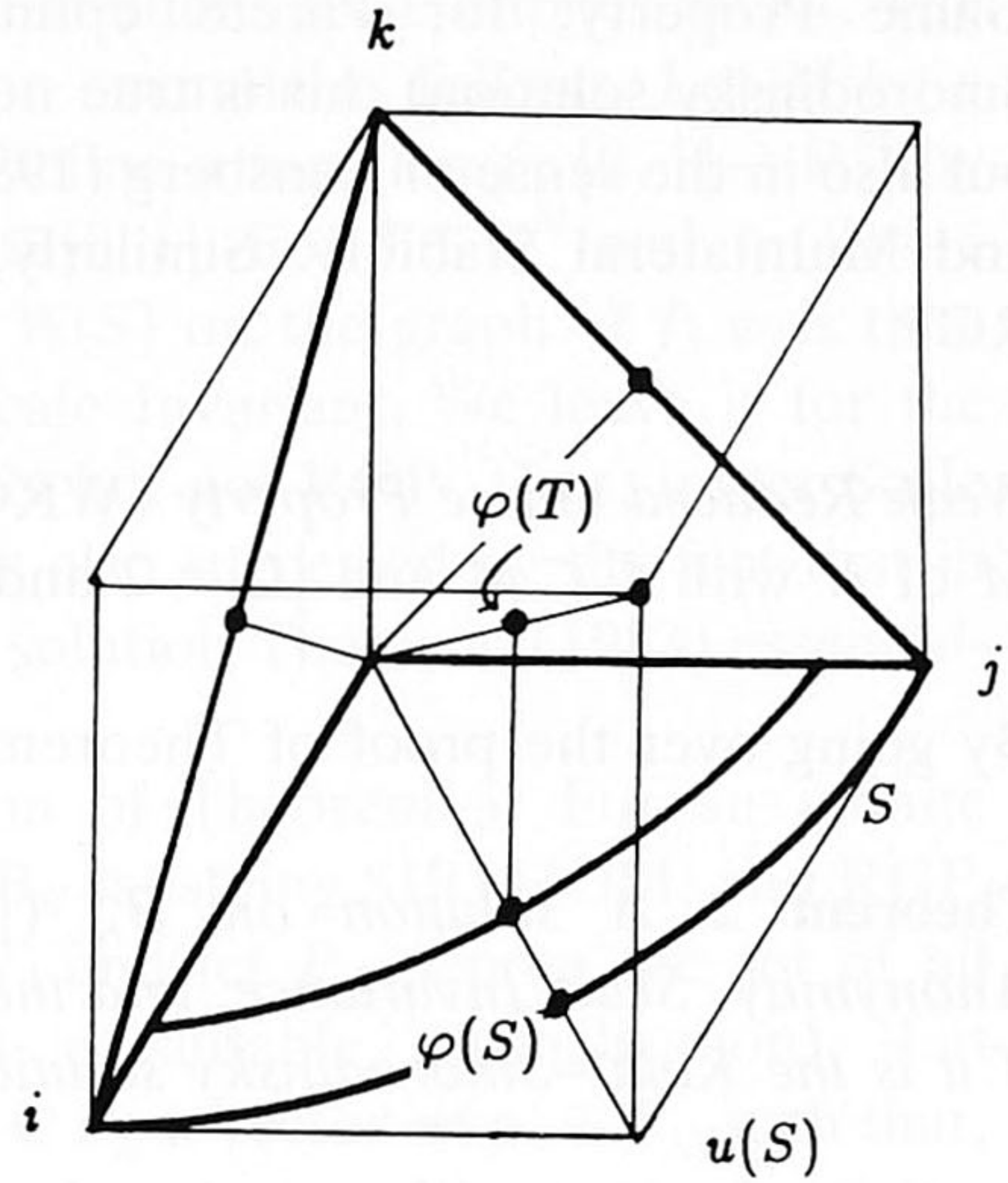


Fig. 2 Proof of Theorem 1.

Theorem 1. A solution on B_N ($|N| > 2$) satisfies Weak Pareto Optimality, Anonymity, Scale Invariance, and the Reduced Game Property, if and only if it is the Kalai–Smorodinsky solution.

Proof. We have already remarked that κ satisfies the four axioms. Let now φ be a solution satisfying the four axioms. We will first prove that if $|M| = 2$ and $S \in B^M$, then $\varphi(S) = \kappa(S)$.

Let $M = \{i, j\}$ and $S \in B^M$ (cf. Fig. 2). By SI, we may assume $u_i(S) = u_j(S) = 1$. Let $k \in N \setminus M$ and

$$T := \text{convex hull}(S \cup \{e^k\}) \subset \mathbb{R}_+^{\{i,j,k\}}$$

where $e_i^k = e_j^k = 0$ and $e_k^k = 1$. By WPO and AN we have

$$\varphi_i(T_{\{i,k\}}) = \varphi_k(T_{\{i,k\}}) = \varphi_j(T_{\{j,k\}}) = \varphi_k(T_{\{j,k\}}) = \frac{1}{2}.$$

By RGP and SI it follows that $\varphi_i(T) = \varphi_j(T)$, and applying RGP and SI again, we obtain $\varphi(S) = \kappa(S)$.

If $|M| = 1$ and $S \in B^M$, then $\varphi(S) = \kappa(S)$ by WPO of φ .

Let now $|M| > 2$ and $S \in B^M$ with (without loss of generality by SI) $u_i(S) = 1$ for every $i \in M$. Let $i, j \in M$, then $\varphi_i(S_{\{i,j\}}) = \varphi_j(S_{\{i,j\}})$ by the above and SI. Hence by RGP: $\varphi_i(S) = \varphi_j(S)$. Since this holds for all $i, j \in M$, we conclude by WPO: $\varphi(S) = \kappa(S)$. \square

Theorem 1 does not hold if there are only two players in the player population N . In that case, any weakly Pareto optimal bargaining solution has the Reduced

Game Property; for Pareto optimal solutions (like the two-player Kalai–Smorodinsky solution) this is true not only in the sense as defined in this paper, but also in the sense of Lensberg (1988). Lensberg distinguishes between Bilateral and Multilateral Stability. Similarly, we introduce the following weakening of RGP.

Weak Reduced Game Property (WRGP): For all non-empty finite subsets L and M of N with $L \subset M$ and $|L| = 2$ and all $S \in B^M$: $\varphi(S_L^{\varphi(S)}) = \varphi(S)_L$.

By going over the proof of Theorem 1, the following result is immediate.

Theorem 2. *A solution on B_N ($|N| > 2$) satisfies Weak Pareto Optimality, Anonymity, Scale Invariance, and the Weak Reduced Game Property, if and only if it is the Kalai–Smorodinsky solution.*

Call a solution φ on B_N *Strongly Individually Rational (SIR)* if $\varphi(S) > 0$ for all non-empty subsets M of N and all $S \in B^M$. Then we have:

Lemma 1. *Let φ be a Strongly Individually Rational and Homogeneous solution on B_N satisfying the Reduced Game Property, and let M be a non-empty finite proper subset of N . Let $S \in B^M$. Then $\varphi(S) \in W(S)$.*

Proof. Take $k \in N \setminus M$, and let $T \in B^{M \cup \{k\}}$ be the convex hull of S and the k th unit vector in $\mathbb{R}_+^{M \cup \{k\}}$. By SIR, $\varphi(T)_M \neq 0$. By RGP,

$$\varphi(\lambda(S, \varphi(T)_M)S) = \varphi(T)_M \in W(\lambda(S, \varphi(T)_M)S).$$

So by HOM, $\varphi(S) \in W(S)$. \square

An immediate consequence of Lemma 1 and Theorem 1 is the following theorem.

Theorem 3. *Let N be infinite. A solution on B_N satisfies Anonymity, Scale Invariance, the Reduced Game Property, and Strong Individual Rationality, if and only if it is the Kalai–Smorodinsky solution.*

The infiniteness of N in Theorem 3 is essential. Consider, for example, the solution φ on $B_{\{1,2,3\}}$, defined by $\varphi(S) := \kappa(S)$ if $S \in B^M$ and $|M| < 3$, and $\varphi(S) := \frac{1}{2}\kappa(S)$ if $S \in B^{\{1,2,3\}}$. This solution satisfies all the axioms in the theorem.

Thomson (1983) uses the following axiom in a characterization of the Kalai–Smorodinsky solution.

Monotonicity with respect to changes in the number of agents (MON): For all non-empty finite subsets $L \subset M$ of N and all $S \in B^L$, $T \in B^M$, if $S = T_L$, then $\varphi(S) \geq \varphi_L(T)$.

It can be verified easily that for Scale Invariant solutions RGP implies MON. The converse, however, is not true. We construct an example as follows. Let M be a non-empty finite subset of N , and let $n \in N$. Define a function $f: [0, 1] \rightarrow \mathbb{R}^M$ by $f_i(t) := t$ if $i \in M$, $i \neq n$ and $f_n(t) := t^2$, for all $t \in [0, 1]$. For $S \in B^M$ with $u_i(S) = 1$ for all $i \in M$, let $\varphi(S)$ be the unique point of $W(S)$ on the graph of f ; φ is then defined on all of B^M by requiring it to be Scale Invariant. We leave it for the reader to verify that this solution satisfies MON but not RGP. Thus, under Scale Invariance, MON is weaker than RGP. This is also suggested by the fact that in his characterization of the Kalai–Smorodinsky solution Thomson (1983) essentially needs an infinite population of agents.

We conclude this section with an extension of Theorem 3. For an infinite population N we will describe all solutions on B_N satisfying SIR, HOM, and RGP. To this end, let i_1 denote the minimum of N , and let P_N denote the set of all vectors $p \in \mathbb{R}_{++}^N$ with $p_{i_1} = 1$ (this is just a suitable normalization). Let $\pi: P_N \rightarrow P_N$ be a map associating with every $p \in P_N$ a vector $\pi(p) \in P_N$ such that, for all $p, \bar{p} \in P_N$ and all finite subsets M of N , if p_M is a positive multiple of \bar{p}_M , then $\pi(p)_M$ is a positive multiple of $\pi(\bar{p})_M$. By Π_N we denote the collection of all such maps π . With $\pi \in \Pi_N$ we associate a bargaining solution φ^π on B_N , as follows. Let $S \in B^M$, where M is a finite subset of N , then $\varphi^\pi(S)$ is the unique point of $W(S)$ of the form $\alpha \pi(p)_M$ ($\alpha \in \mathbb{R}_{++}$), where p_M is a multiple of the utopia point of S . We call such a solution φ^π a *generalized proportional solution*. For π being identity, φ^π is the Kalai–Smorodinsky solution. If π is the constant map assigning the vector with all ones to every p , then φ^π is the *egalitarian solution*.

It is easy to verify that for every $\pi \in \Pi_N$, the solution φ^π satisfies SIR and HOM. RGP follows by the special condition imposed on the map π . Conversely, if φ is a solution satisfying SIR, HOM, and RGP, then by Lemma 1 it is Weakly Pareto Optimal. Then, the proof of Theorem 1 can be adapted to show that φ is of the form φ^π . Summarizing, we have:

Theorem 4. *Let N be infinite. A solution on B_N satisfies Homogeneity, the Reduced Game Property, and Strong Individual Rationality, if and only if it is a generalized proportional solution.*

Also Theorem 1 can be extended in a similar spirit, but if SIR is not required, then so-called ‘weighted hierarchies’ (cf. Peters, 1992, p. 19) may arise in the population, due to the possibility of players with zeros in the vector $p \in \Pi_N$. The description of the associated solutions is rather technical and therefore omitted.

3. Concluding remarks

Krishna and Serrano (1990) present a non-cooperative implementation of the Nash bargaining solution based on Lensberg’s (1988) multilateral stability axiom.

Similarly, in an earlier version of this paper¹ a non-cooperative game is described of which the unique subgame perfect equilibrium outcome is given by applying the Kalai–Smorodinsky solution. The subgames of that non-cooperative game correspond to the reduced games defined in the previous section. An earlier non-cooperative implementation of the Kalai–Smorodinsky solution is provided by Moulin (1984).

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¹ Available from the authors.